SEMICLASSICAL GRAVITY THEORY AND QUANTUM FLUCTUATIONS

Chung-I Kuo and L. H. Ford

Institute of Cosmology Department of Physics and Astronomy Tufts University Medford, Massachusetts 02155

We discuss the limits of validity of the semiclassical theory of gravity in which a classical metric is coupled to the expectation value of the stress tensor. It is argued that this theory is a good approximation only when the fluctuations in the stress tensor are small. We calculate a dimensionless measure of these fluctuations for a scalar field on a flat background in particular cases, including squeezed states and the Casimir vacuum state. It is found that the fluctuations are small for states which are close to a coherent state, which describes classical behavior, but tend to be large otherwise. We find in all cases studied that the energy density fluctuations are large whenever the local energy density is negative. This is taken to mean that the gravitational field of a system with negative energy density, such as the Casimir vacuum, is not described by a fixed classical metric but is undergoing large metric fluctuations. We propose an operational scheme by which one can describe a fluctuating gravitational field in terms of the statistical behavior of test particles. For this purpose we obtain an equation of the form of the Langevin equation used to describe Brownian motion.

Typeset Using *REVTEX*

I. INTRODUCTION

A natural proposal to describe the gravitational field of a quantum system is the semiclassical theory in which the expectation value of the stress tensor is the source. The semiclassical Einstein equation is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N \langle T_{\mu\nu} \rangle.$$
(1.1)

This theory is almost certain to fail at the Planck scale, where the quantum nature of gravity becomes important. However, it can also fail far away from the Planck scale if the fluctuations in the stress tensor become important. This was discussed some time ago by one of us [1] and illustrated by the problem of graviton emission by a box of particles in a general quantum state. It was found that the semiclassical theory gives reliable results when the fluctuations in the stress tensor are not too large, that is, when

$$\langle T_{\alpha\beta}(x) T_{\mu\nu}(y) \rangle \approx \langle T_{\alpha\beta}(x) \rangle \langle T_{\mu\nu}(y) \rangle.$$
 (1.2)

However, for quantum states in which the energy density fluctuations are large, the semiclassical theory based upon Eq.(1.1) cannot be trusted.

In the present paper, we will further explore the issue of the limits of validity of the semiclassical theory. Particular attention will be paid to quantum states in which the expectation value of the local energy density can be negative. In Sec. II some aspects of such states will be reviewed, partly for the purpose of establishing notation and results which will be used later. In Sec. III the range of validity of the semiclassical theory will be probed in particular flat space examples, including squeezed states and the Casimir effect. In Sec. IV a proposal will be made as to how to describe the gravitational field of a system in which the energy density fluctuations are large. We suggest that a statistical description of the Brownian-like motion of test particles in such a gravitational field yields all of the information which is available about the system. Finally, in Sec. V the results will be summarized and discussed.

We adopt the convention $c = \hbar = 1$ and a spacelike metric $\eta_{\mu\nu} = diag[-1, 1, 1, 1]$ throughout this paper.

II. QUANTUM STATES OF NEGATIVE ENERGY DENSITY

Although all known forms of classical matter have non-negative energy density, it is not so in quantum field theory. A general state can be a superposition of number eigenstates and may have a negative expectation value of energy density in certain spacetime regions due to coherence effects, thus violating the weak energy condition [2]. If there were no constraints on the extent of the violation of the weak energy condition, several dramatic and disturbing effects could arise. These include the breakdown of the second law of thermodynamics [3], of cosmic censorship [4], and of causality citeMorris88. There are, however, two possible reasons as to why these effects will not actually arise. The first is the existence of constraints on the magnitude and the spatial or temporal extent of the negative energy [3,4,6]. The second is that the semiclassical theory of gravity may not be applicable to systems in which the energy density is negative. This latter possibility will be the main topic to be investigated in this paper.

Let us consider a massless, minimally coupled scalar field for which the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \eta_{\mu\nu} (\partial^{\mu} \phi) (\partial^{\nu} \phi).$$
(2.1)

The stress tensor is

$$T_{\mu\nu} = \Pi(\partial_0 \phi) - \eta_{\mu\nu} \mathcal{L} = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}\eta_{\mu\nu}(\partial_\sigma \phi)(\partial^\sigma \phi), \qquad (2.2)$$

where $\Pi = \partial_0 \phi$ is the conjugate momentum of ϕ . Variation with respect to ϕ gives the dynamical equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi(x) \equiv \partial_\mu \partial^\mu \phi(x) = 0.$$
(2.3)

The quantum field operator ϕ may be expanded in mode functions as

$$\phi = \sum_{\mathbf{k}} (a_{\mathbf{k}} f_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}^{*}), \qquad (2.4)$$

where

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}, \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger}] = 0, \qquad (2.5)$$

and the mode function is a solution of the dynamical equation

$$f_{\mathbf{k}} = (2L^{3}\omega)^{-1/2}e^{ik_{\sigma}x^{\sigma}} = (2L^{3}\omega)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}.$$
(2.6)

Here we have assumed periodic boundary conditions in a 3-dimensional box of side L. The sum $\sum_{\mathbf{k}}$ is transformed to $\frac{L^3}{(2\pi)^3} \int d^3k$ in the continuum limit. The mode functions are normalized on a spacelike hypersurface under the scalar product

$$(\phi_1, \phi_2) \equiv -i \int d^{n-1}x \{ \phi_1(x) \partial_0 \phi_2^*(x) - (\partial_0 \phi_1(x)) \phi_2^*(x) \},$$
(2.7)

so that

$$(f_{\mathbf{k}}, f_{\mathbf{k}'}) = \delta_{\mathbf{k}\mathbf{k}'}.$$
(2.8)

We will here consider states where a single mode is excited. Such a quantum state takes the form

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \qquad (2.9)$$

where $|n\rangle$ is a number eigenstate with *n* particles in mode **k** and the c_n are any coefficients such that $\sum_{n=0}^{\infty} |c_n|^2 = 1$. The normal ordered expectation value of the stress tensor is

$$\langle : T_{\alpha\beta}(x) : \rangle = \langle \Psi | : T_{\alpha\beta} : |\Psi \rangle$$

= $\sum_{n=0}^{\infty} (2n|c_n|^2 T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] + n^{1/2}(n-1)^{1/2} c_n c_{n-2}^* T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}]$
+ $n^{1/2}(n-1)^{1/2} c_n^* c_{n-2} T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*]),$ (2.10)

where $T_{\alpha\beta}[g,h]$ is the bilinear form

$$T_{\mu\nu}[g,h](x) = (\partial_{\mu}g)(\partial_{\nu}h) - \frac{1}{2}\eta_{\mu\nu}(\partial_{\sigma}g)(\partial^{\sigma}h).$$
(2.11)

The different bilinear forms arising from the mode functions are

$$T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] = -\mathcal{K}_{\alpha\beta}e^{2i\theta}, \qquad (2.12)$$

$$T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}] = T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] = \mathcal{K}_{\alpha\beta}, \qquad (2.13)$$

and

$$T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*] = -\mathcal{K}_{\alpha\beta}e^{-2i\theta}, \qquad (2.14)$$

where $\theta \equiv k_{\rho}x^{\rho}$ and $\mathcal{K}_{\alpha\beta} \equiv (k_{\alpha}k_{\beta} - \frac{1}{2}\eta_{\alpha\beta}k_{\rho}k^{\rho})/(2\omega L^{3})$. For the massless case $k_{\rho}k^{\rho} = 0$, so $\mathcal{K}_{\alpha\beta} = \frac{k_{\alpha}k_{\beta}}{(2\omega L^{3})}$. Notice that $:T_{\alpha\beta}:=T_{\alpha\beta}-\langle 0|T_{\alpha\beta}|0\rangle$; the normal ordered stress tensor in flat spacetime is the renormalized result obtained by subtracting the Minkowski vacuum expectation value.

The quantum coherence effects which produce negative energy densities can be easily illustrated by the state composed of two particle number eigenstates

$$|\Psi\rangle \equiv \frac{1}{\sqrt{1+\epsilon^2}} (|0\rangle + \epsilon |2\rangle), \qquad (2.15)$$

where $|0\rangle$ is the vacuum state satisfying $a|0\rangle = 0$, and $|2\rangle = \frac{1}{\sqrt{2}}(a^{\dagger})^2|0\rangle$ is the two particle state. Here we take ϵ , the relative amplitude of the two states, to be real for simplicity. For this state,

$$\langle : T_{\alpha\beta}(x) : \rangle = \langle \Psi | : T_{\alpha\beta}(x) : | \Psi \rangle$$

$$= \frac{\epsilon}{1+\epsilon^2} \left\{ \sqrt{2} (T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*]) + 2\epsilon (T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] + T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}]) \right\}$$

$$= \frac{\mathcal{K}_{\alpha\beta}\epsilon}{1+\epsilon^2} (2\epsilon - \sqrt{2}\cos(2\theta)).$$

$$(2.16)$$

Obviously the energy density can be positive or negative depending on the value of ϵ and the spacetime-dependent phase $\theta \equiv k_{\rho}x^{\rho}$. We also observe that the negative contribution comes from the cross term. For a general state which is a linear combination of N particle number eigenstates, the number of cross terms will increase as N^2 . Therefore, for a general quantum state, the occurrence of a negative energy density is very probable. However, the total energy is always non-negative. We will come back to this simple illustrative example when discussing the breakdown of the semiclassical gravity theory later.

III. LIMITS OF THE SEMICLASSICAL THEORY

To obtain a criterion of the validity of semiclassical gravity theory, we may recall the investigation in Ref. [1]. There the energy flux of gravitational radiation in linearized gravity produced by a matter field was calculated both in the semiclassical theory and in a linear quantum gravity theory. In the semiclassical theory based upon Eq.(1.1), the flux depends on products of expectation values of stress tensor operators, whereas in a theory in which the metric perturbations are quantized, it depends upon the corresponding products of expectation values. For the particular case of a single-mode coherent state, the product of expectation values is the same as the expectation value of products of stress tensors. This can be understood from the fact that a coherent state corresponds to a classical field excitation, for which one would expect the semiclassical theory to be a good approximation.

We propose that the extent to which the semiclassical approximation is violated can be measured by the dimensionless quantity

$$\Delta_{\alpha\beta\mu\nu}(x,y) \equiv \left| \frac{\langle : T_{\alpha\beta}(x) T_{\mu\nu}(y) : \rangle - \langle : T_{\alpha\beta}(x) : \rangle \langle : T_{\mu\nu}(y) : \rangle}{\langle : T_{\alpha\beta}(x) T_{\mu\nu}(y) : \rangle} \right|.$$
(3.1)

This quantity is a dimensionless measure of the stress tensor fluctuations. (Note that it is not a tensor, but rather the ratio of tensor components.) If its components are always small compared to unity, then these fluctuations are small and we expect the semiclassical theory to hold. However, the numerous components and the dependence upon two spacetime points make this a rather cumbersome object to study. For simplicity, we will concentrate upon the coincidence limit, $x \to y$, of the purely temporal component of the above quantity, that is

$$\Delta(x) \equiv \left| \frac{\langle : T_{00}^2(x) : \rangle - \langle : T_{00}(x) : \rangle^2}{\langle : T_{00}^2(x) : \rangle} \right|.$$
(3.2)

The local energy density fluctuations are small when $\Delta \ll 1$, which we take to be a measure of the validity of the semiclassical theory. Note that we have used normal ordering with respect to the Minkowski vacuum state to define the various operators.

It is not difficult to see why the semiclassical gravity theory is not expected to be valid when the energy fluctuations are large. Suppose we have a quantum state which is a superposition of two states, each of which describes a distinct macroscopic matter configuration, e.g., (1) the presence of a 1000 kg mass, or (2) the absence of this mass. Clearly, a measurement of the gravitational field should reveal either (1) the field of a 1000 kg mass, or (2) no field, each with 50% probability. But the semiclassical field equations predict finding the field of a 500 kg mass with 100% probability.

Now we will discuss some specific examples involving the massless scalar field in flat spacetime. First we consider the general quantum state $|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, for which a single mode is excited. The correlation function for the stress tensor can be divided into three parts,

$$\langle :T_{\alpha\beta} T_{\mu\nu} : \rangle = \langle \Psi | :T_{\alpha\beta} T_{\mu\nu} : |\Psi\rangle = P_0 + P_2 + P_4.$$
(3.3)

The contribution of the diagonal terms is

$$P_{0} = \sum_{n=0}^{\infty} |c_{n}|^{2} (n-1)^{1/2} n (n+1)^{1/2} (T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}]).$$
(3.4)

That of the matrix elements between $|n\rangle$ and $|n+2\rangle$ is

$$P_{2} = \sum_{n=0}^{\infty} c_{n} c_{n+2}^{*} n (n+1)^{1/2} (n+2)^{1/2} (T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}^{*}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}] T_{\mu\nu}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}]) + c.c., \qquad (3.5)$$

whereas that between $|n\rangle$ and $|n+4\rangle$ is

$$P_4 = \sum_{n=0}^{\infty} c_n c_{n+4} (n+4)^{1/2} (n+3)^{1/2} (n+2)^{1/2} (n+1)^{1/2} T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*] T_{\mu\nu}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*] + c.c. , \quad (3.6)$$

where c.c. denotes the complex conjugate. The effect of the normal ordering has been to remove the contribution of the n = 0 term from the diagonal part, P_0 .

A. The Vacuum + Two Particle State

Now we go back to the simple case of a state which is a superposition of the vacuum and a two particle eigenstate, Eq. (2.15). In this case we find

$$\langle : T_{\alpha\beta}(x) T_{\mu\nu}(x) : \rangle = \frac{2\epsilon^2}{(1+\epsilon^2)^2} \left(T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}^*, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^*, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] T_{\mu\nu}[f_{\mathbf{k}}^*, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] T_{\mu\nu}[f_{\mathbf{k}}, f_{\mathbf{k}}^*] + T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] T_{\mu\nu}[f_{\mathbf{k}}^*, f_{\mathbf{k}}^*] \right) = \mathcal{K}_{\alpha\beta} \mathcal{K}_{\mu\nu} \frac{12\epsilon^2}{(1+\epsilon^2)^2}.$$

$$(3.7)$$

From this result, Eq. (2.16), and the fact that $\mathcal{K}_{00} = \frac{\omega}{2L^3}$, we find that

$$\Delta(x) = \frac{10\epsilon + \sqrt{2}\cos(2\theta)}{12\epsilon}.$$
(3.8)

From Eq.(2.16), it follows that the condition for the expectation value of energy density to be negative is $cos(2\theta) > \sqrt{2}\epsilon$. In this case we have $\Delta(x) > 1$. This indicates when the energy density is negative, the energy density fluctuations are large and the semiclassical theory is not a good approximation.

B. Squeezed States

Squeezed states of light have been extensively investigated recently in quantum optics and have been experimentally realized [7]. Squeezing reduces the quantum uncertainty in one variable with a corresponding increase in that of its conjugate variable. Squeezed states form a two parameter family of quantum states, which include both coherent states and states with negative energy density as different limits.

A general squeezed state for a single mode can be expressed as [8]

$$|\alpha,\zeta\rangle = D(\alpha)\,S(\zeta)\,|0\rangle,\tag{3.9}$$

where $D(\alpha)$ is the displacement operator

$$D(\alpha) \equiv exp(\alpha a^{\dagger} - \alpha^* a) = e^{-|\alpha|^2/2} e^{\alpha a^{\dagger}} e^{-\alpha^* a}$$
(3.10)

and $S(\zeta)$ is the squeeze operator

$$S(\zeta) \equiv exp[\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^{\dagger})^2].$$
(3.11)

Here

$$\alpha = se^{i\gamma} \tag{3.12}$$

and

$$\zeta = r e^{i\delta}.\tag{3.13}$$

are arbitrary complex numbers. The displacement and squeeze operators satisfy the relations

$$D^{\dagger}(\alpha) a D(\alpha) = a + \alpha, \qquad (3.14)$$

$$D^{\dagger}(\alpha) a^{\dagger} D(\alpha) = a^{\dagger} + \alpha^*, \qquad (3.15)$$

$$S^{\dagger}(\zeta) a S(\zeta) = a \cosh r - a^{\dagger} e^{i\delta} \sinh r, \qquad (3.16)$$

and

$$S^{\dagger}(\zeta) a^{\dagger} S(\zeta) = a^{\dagger} \cosh r - a e^{-i\delta} \sinh r.$$
(3.17)

Using the above formulae, the expectation value of the renormalized stress tensor is easily obtained:

$$\langle \alpha, \zeta |: T_{\alpha\beta}(x) : |\alpha, \zeta \rangle = \langle 0 | S^{\dagger}(\zeta) D^{\dagger}(\alpha) (a^{2}T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] + a^{\dagger}a (T_{\alpha\beta}[f_{\mathbf{k}}, f_{\mathbf{k}}] + T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}]) + (a^{\dagger})^{2}T_{\alpha\beta}[f_{\mathbf{k}}^{*}, f_{\mathbf{k}}^{*}]) D(\alpha) S(\zeta) |0\rangle$$

$$= 2\mathcal{K}_{\alpha\beta} \{\sinh r \cosh r \cos(2\theta + \delta) + \sinh^{2}r$$

$$+ s^{2}[1 - \cos 2(\theta + \gamma)]\}$$

$$(3.18)$$

Similarly, the expectation value of the squared stress tensor is

$$\langle \alpha, \zeta |: T_{\alpha\beta}(x)T_{\mu\nu}(x): |\alpha, \zeta \rangle = 2\mathcal{K}_{\alpha\beta} \,\mathcal{K}_{\mu\nu} \Big(s^4 [\cos 4(\theta + \gamma) - 4\cos 2(\theta + \gamma) + 3] \\ + 3s^2 \Big\{ 2\sinh r \cosh r [2\cos(2\theta + \delta + 2\gamma) - \cos(4\theta + \delta + 2\gamma)] \\ + 4\sinh^2 r (\cos 2\gamma - \cos 2\theta) - \cos(\delta + 2\gamma) \Big\} \\ + 3\sinh^2 r [\cosh^2 r \cos(4\theta + 2\delta) + 3 - 4\cos 2\theta] \Big).$$
(3.19)

The degree of squeezing may be measured by the squeeze parameter $|\zeta| = r$. When $\zeta = 0$, the squeezed states reduce to coherent states. In this particular case, the expectation value of a product of stress tensors is equal to the corresponding product of expectation values [1]:

$$\langle : T_{\alpha\beta}(x) T_{\mu\nu}(y) : \rangle = \langle : T_{\alpha\beta}(x) : \rangle \langle : T_{\mu\nu}(y) : \rangle, \qquad (3.20)$$

and hence

$$\Delta_{\alpha\beta\mu\nu}(x,y) = \Delta = 0. \tag{3.21}$$

This result is consistent with the interpretation of coherent states as the quantum states which describe classical field excitations. By the criterion which we have adopted, the semiclassical gravity theory is a good approximation for coherent states.

The opposite limit from a coherent state is the case where $\alpha = 0$, known as a squeezed vacuum state. Such a state is not, of course, the vacuum state so long as $\zeta \neq 0$, but rather a superposition of states containing even numbers of particles. Squeezed vacuum states have a particular physical interest because they are the states which result from quantum particle creation processes. Such states of the electromagnetic field have been generated in the laboratory using nonlinear optical media, and have been the topic of much interest in quantum optics in recent years.

In a squeezed vacuum, $\alpha = 0$, we may also take $\delta = 0$, as this is simply a choice of phase, and write

$$\langle \alpha, \zeta |: T_{\alpha\beta}(x): |\alpha, \zeta \rangle = 2\mathcal{K}_{\alpha\beta} \sinh r [\cosh r \cos(2\theta) + \sinh r]. \tag{3.22}$$

Here r = 0 corresponds to the vacuum state and hence gives a vanishing expectation value for the stress tensor. For $r \neq 0$, the squeezed vacuum state exhibits negative energy densities. That is, for fixed r as θ varies from 0 to 2π (either the spatial position changing by one wavelength at fixed time, or the time varying through one period at fixed position), the energy density becomes negative during part of the cycle.

For the case $\delta = 0$ and $\alpha = 0$, the expectation value of the squared stress tensor is

$$\langle \alpha, \zeta |: T_{\alpha\beta}(x)T_{\mu\nu}(x): |\alpha, \zeta \rangle = 2\mathcal{K}_{\alpha\beta}\,\mathcal{K}_{\mu\nu}\sinh^2 r [2\cosh^2 r \cos 4\theta - 8\sinh r \cosh r \cos 2\theta + 3(\sinh^2 r + \cosh^2 r)].$$
(3.23)

As required, for the vacuum state (r = 0) this quantity vanishes.

From the above results, we can form the quantity Δ . However, the analytical expressions are not particularly transparent. Our primary concern is whether or not $\Delta \ll 1$, which is best determined by numerical evaluation of Δ using Eqs. (3.2), (3.18), and (3.19). The figures illustrate the results. For states which are sufficiently close to coherent states, i.e. $r \ll |\alpha|$, we do indeed find that $\Delta \ll 1$. However, as the magnitude of the squeeze parameter r increases relative to that of $|\alpha|$, we find that Δ increases. By the point that the state is sufficiently squeezed to have $\rho = \langle : T_{00} : \rangle < 0$, we always have that Δ is at least of order unity. Thus squeezed states for which the energy density is negative exhibit large energy density fluctuations.

C. The Casimir Vacuum

One of the most astonishing predictions of quantum field theory is the Casimir effect [9], in which the vacuum energy density of the quantized electromagnetic field between two parallel perfectly conducting plates is negative. We will consider the Casimir effect for a massless, minimally coupled scalar field. The expectation value of the stress tensor can be expressed as

$$\langle T_{\mu\nu}(x)\rangle = \frac{1}{2}\lim_{x'\to x} (\nabla_{\mu'}\nabla_{\nu} + \nabla_{\mu}\nabla_{\nu'} - g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha'}) G(x, x'), \qquad (3.24)$$

where the Hadamard elementary function G(x, x') is

$$G(x, x') = \frac{1}{2} \langle \phi(x)\phi(x') + \phi(x')\phi(x) \rangle.$$
(3.25)

This quantity is formally infinite and is renormalized by replacing G(x, x') by the renormalized Green's function

$$G_R(x, x') = G(x, x') - G_0(x, x'), \qquad (3.26)$$

where $G_0(x, x')$ is the Minkowski space Green's function, i.e., that in the absence of boundaries. The resulting finite stress energy is the expectation value in the Casimir vacuum of $T_{\mu\nu}$ normal ordered with respect to the Minkowski vacuum:

$$\langle : T_{\mu\nu}(x) : \rangle_C = \frac{1}{2} \lim_{x' \to x} (\nabla_{\mu'} \nabla_{\nu} + \nabla_{\mu} \nabla_{\nu'} - g_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha'}) G_R(x, x').$$
(3.27)

For the purpose of illustration, we will consider the particular boundary condition of periodicity in the z-direction with periodicity length L. In this case, the Green's function can be expressed as an image sum:

$$G(x, x') = \sum_{n = -\infty}^{\infty} G_n(\sigma_n(x, x')), \qquad (3.28)$$

where

$$G_n(\sigma_n(x, x')) = \frac{1}{4\pi^2 \sigma_n}.$$
 (3.29)

The geodesic distance σ for different *n* is

$$\sigma_n(x, x') = -(t - t')^2 + (\mathbf{r} - \mathbf{r}'_n)^2, \qquad (3.30)$$

where

$$x = (t, \mathbf{r}), \quad x' = (t', \mathbf{r}'), \quad \mathbf{r}'_n = (x', y', z' - nL).$$
 (3.31)

The renormalized Green's function $G_R(x, x')$ is simply given by Eq. (3.28) with the n = 0 term omitted.

Thus we find

$$\langle :T_{\mu\nu}(x): \rangle_C = \frac{1}{8\pi^2} \lim_{x' \to x} \sum_{n=-\infty}^{\infty} (\partial_{\mu'} \partial_{\nu} + \partial_{\mu} \partial_{\nu'} + \eta_{\mu\nu} \eta^{\rho\sigma} \partial_{\rho} \partial_{\sigma'}) \sigma_n^{-1}$$
$$= -\frac{\zeta(4)}{\pi^2 L^4} diag[1, -1, -1, 3], \qquad (3.32)$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function, and we have excluded the n = 0 term, since it is just the contribution from the Minkowski vacuum. Since $\zeta(4) = \frac{\pi^4}{90}$,

$$\langle :T_{00}(x): \rangle = -\frac{\pi^2}{90 L^4}.$$
 (3.33)

We now proceed to define the stress tensor correlation function as before by interpreting $: T_{\alpha\beta}(x) T_{\mu\nu}(y)$: as being normal ordered with respect to the Minkowski vacuum state. Its expectation value is obtained by differentiation of the corresponding four-point function

$$\langle : T_{\alpha\beta}(x)T_{\mu\nu}(y): \rangle = \lim_{\substack{x_1,x_2 \to x \\ x_3,x_4 \to y}} \left(\eta_{\alpha\lambda}\eta_{\beta\gamma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\lambda\gamma} \right) \left(\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} \right) \times \\ \partial_1^{\lambda} \partial_2^{\gamma} \partial_3^{\rho} \partial_4^{\sigma} \langle : \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \rangle_C.$$

$$(3.34)$$

This four-point function is the expectation value of an operator normal ordered with respect to the Minkowski vacuum, but whose expectation value is taken in the Casimir vacuum state. We have in fact dealt with similar quantities in our calculation of the mean stress tensor, namely $\langle : T_{\mu\nu} : \rangle_C$ itself and the renormalized Green's function

$$G_R(x, x') = \frac{1}{2} \langle : \phi(x)\phi(x') + \phi(x')\phi(x) : \rangle_C.$$
(3.35)

The evaluation of the expectation value of the four-point function can be achieved with the aid of Wick's theorem. Let $\phi_i = \phi(x_i)$. The quartic product may be expressed as

$$\phi_{1}\phi_{2}\phi_{3}\phi_{4} =: \phi_{1}\phi_{2}\phi_{3}\phi_{4}: +: \phi_{1}\phi_{2}: \langle\phi_{3}\phi_{4}\rangle_{M} +: \phi_{1}\phi_{3}: \langle\phi_{2}\phi_{4}\rangle_{M} +: \phi_{1}\phi_{4}: \langle\phi_{2}\phi_{3}\rangle_{M} + : \phi_{2}\phi_{3}: \langle\phi_{1}\phi_{4}\rangle_{M} +: \phi_{2}\phi_{4}: \langle\phi_{1}\phi_{3}\rangle_{M} +: \phi_{3}\phi_{4}: \langle\phi_{1}\phi_{2}\rangle_{M} + \langle\phi_{1}\phi_{2}\rangle_{M} \langle\phi_{3}\phi_{4}\rangle_{M} + \langle\phi_{1}\phi_{3}\rangle_{M} \langle\phi_{2}\phi_{4}\rangle_{M} + \langle\phi_{1}\phi_{4}\rangle_{M} \langle\phi_{2}\phi_{3}\rangle_{M},$$
(3.36)

where $\langle \rangle_M$ is the expectation value in the Minkowski vacuum. However, it may equally well be written as

$$\phi_{1}\phi_{2}\phi_{3}\phi_{4} = N_{C}(\phi_{1}\phi_{2}\phi_{3}\phi_{4}) + N_{C}(\phi_{1}\phi_{2})\langle\phi_{3}\phi_{4}\rangle_{C} + N_{C}(\phi_{1}\phi_{3})\langle\phi_{2}\phi_{4}\rangle_{C} + N_{C}(\phi_{1}\phi_{4})\langle\phi_{2}\phi_{3}\rangle_{C} + N_{C}(\phi_{2}\phi_{3})\langle\phi_{1}\phi_{4}\rangle_{C} + N_{C}(\phi_{2}\phi_{4})\langle\phi_{1}\phi_{3}\rangle_{C} + N_{C}(\phi_{3}\phi_{4})\langle\phi_{1}\phi_{2}\rangle_{C} + \langle\phi_{1}\phi_{2}\rangle_{C}\langle\phi_{3}\phi_{4}\rangle_{C} + \langle\phi_{1}\phi_{3}\rangle_{C}\langle\phi_{2}\phi_{4}\rangle_{C} + \langle\phi_{1}\phi_{4}\rangle_{C}\langle\phi_{2}\phi_{3}\rangle_{C},$$
(3.37)

where N_C denotes normal ordering with respect to the Casimir vacuum. Recall that Wick's theorem applies to any quantum state $|\psi\rangle$ for which there is a decomposition of the field operator into positive and negative frequency parts, $\phi = \phi^+ + \phi^-$, so that $\phi^+ |\psi\rangle = \langle \psi | \phi^- = 0$. Both the Minkowski and Casimir vacua satisfy this condition. We now wish to equate the two expressions above, solve for : $\phi_1 \phi_2 \phi_3 \phi_4$:, and take its expectation value in the Casimir vacuum. If we use such relations as $\langle N_C(\phi_1\phi_2)\rangle_C = 0$ and

$$\langle \phi_1 \phi_2 \rangle_M = \langle \phi_1 \phi_2 \rangle_C - \langle N_M(\phi_1 \phi_2) \rangle_C, \qquad (3.38)$$

we obtain the result

$$\langle : \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \rangle_C = \langle : \phi(x_1)\phi(x_2) : \rangle_C \langle : \phi(x_3)\phi(x_4) : \rangle_C + \langle : \phi(x_1)\phi(x_3) : \rangle_C \langle : \phi(x_2)\phi(x_4) : \rangle_C + \langle : \phi(x_1)\phi(x_4) : \rangle_C \langle : \phi(x_2)\phi(x_3) : \rangle_C.$$
(3.39)

Using this result, we may express the expectation value of the squared energy density as

$$\langle : T_{00}{}^{2}(x) : \rangle_{C} = \lim_{x_{1}, x_{2}, x_{3}, x_{4} \to x} \left(\delta_{0\lambda} \, \delta_{0\gamma} - \frac{1}{2} \eta_{\lambda\gamma} \right) \left(\delta_{0\rho} \, \delta_{0\sigma} - \frac{1}{2} \eta_{\rho\sigma} \right) \partial_{1}^{\lambda} \, \partial_{2}^{\gamma} \, \partial_{3}^{\rho} \, \partial_{4}^{\sigma} \times \left(\left\langle : \phi(x_{1})\phi(x_{2}) : \right\rangle \left\langle : \phi(x_{3})\phi(x_{4}) : \right\rangle + \left\langle : \phi(x_{1})\phi(x_{3}) : \right\rangle \left\langle : \phi(x_{2})\phi(x_{4}) : \right\rangle + \left\langle : \phi(x_{1})\phi(x_{4}) : \right\rangle \left\langle : \phi(x_{2})\phi(x_{3}) : \right\rangle \right).$$

$$= \left\langle : T_{00} : \right\rangle_{C}^{2} + \frac{1}{2} \left[\left\langle : \dot{\phi}^{2} : \right\rangle_{C}^{2} + \left\langle : (\phi_{,x})^{2} : \right\rangle_{C}^{2} + \left\langle : (\phi_{,y})^{2} : \right\rangle_{C}^{2} + \left\langle : (\phi_{,z})^{2} : \right\rangle_{C}^{2} \right]. \tag{3.40}$$

Let $\rho = \langle : T_{00} : \rangle_C$ be the Casimir energy density and let $p_i = \langle : T_{ii} : \rangle_C$ be the pressure in the *i*-direction. Then for the massless scalar field we have

$$\langle :\dot{\phi}^2: \rangle_C^2 = \frac{1}{2}(\rho + p_1 + p_2 + p_3)^2$$
 (3.41)

and, for example,

$$\langle :(\phi_{,x})^2 : \rangle_C^2 = \frac{1}{2}(\rho + p_1 - p_2 - p_3)^2.$$
 (3.42)

From these relations, we may write

$$\Delta' \equiv \frac{\langle :T_{00}^{2}(x): \rangle - \rho^{2}}{\rho^{2}} = \frac{1}{8} [(\rho + \xi_{1} + \xi_{2} + \xi_{3})^{2} + (\rho + \xi_{1} - \xi_{2} - \xi_{3})^{2} + (\rho - \xi_{1} + \xi_{2} - \xi_{3})^{2} + (\rho - \xi_{1} - \xi_{2} + \xi_{3})^{2}], \qquad (3.43)$$

where $\xi_i = p_i / \rho$. The quantity Δ' is related to Δ by

$$\Delta' = \frac{\Delta}{1 - \Delta}, \qquad \Delta = \frac{\Delta'}{1 + \Delta'}, \qquad (3.44)$$

and is an equivalent measure of the scale of the energy density fluctuations.

We may obtain lower bounds on Δ' and Δ . The minimum value of the right hand side of Eq. (3.43) is 1/2 and occurs when $\xi_1 = \xi_2 = \xi_3 = 0$. Thus we have that

$$\Delta' \ge \frac{1}{2}, \qquad \Delta \ge \frac{1}{3}. \tag{3.45}$$

This means that the dimensionless Casimir energy density fluctuations are always at least of order unity. Note that this bound is independent of the details of the geometry of the boundaries. For the particular case of periodicity in one spatial direction, we have from Eq. (3.32) that $\xi_1 = \xi_2 = -1$ and $\xi_3 = 3$, so that $\Delta' = 6$ and $\Delta = 6/7$. The essential point here is that neither measure of the fluctuations is small, and hence we conclude that the gravitational field due to Casimir energy is not described by a fixed classical metric. That the Casimir force is a fluctuating force, and that Casimir's calculation yields the mean value of that force, have been emphasized by Barton [10].

IV. METRIC FLUCTUATIONS AND TEST PARTICLES

The properties of spacetime can be probed by test particles which follow geodesics in a classical gravitational field. When the gravitational field is described by a classical metric, we can use test particles as an operational probe of the geometry. Our problem is now to give a meaning to the gravitational field of a fluctuating source, such as the Casimir vacuum. We propose to replace the description in which the test particles follow fixed trajectories by a statistical description in which one only attempts to compute average quantities such as the mean squared velocity of an ensemble of test particles. This is the approach which is used to describe Brownian motion by means of a Langevin equation. The basic idea is that the particle is subjected both to a classical force and to a fluctuating force. For nonrelativistic motion, its equation of motion may be written as

$$m\frac{d\mathbf{v}(x)}{dt} = \mathbf{F}_c(x) + \mathbf{F}(x), \qquad (4.1)$$

where m is the test particle mass, \mathbf{F}_c is the classical force, and \mathbf{F} is the fluctuating force. The solution of this equation is

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \frac{1}{m} \int_{t_0}^t [\mathbf{F}_c(t') + \mathbf{F}(t')] dt'.$$
(4.2)

We assume that the fluctuating force averages to zero, $\langle \mathbf{F} \rangle = 0$, but that quantities quadratic in \mathbf{F} do not. Thus the mean squared velocity, averaged over an ensemble of test particles is, for the case that $\mathbf{v}(t_0) = 0$,

$$\langle \mathbf{v}^2 \rangle = \frac{1}{m^2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \left[\mathbf{F}_c(t_1) \, \mathbf{F}_c(t_2) + \langle \mathbf{F}(t_1) \, \mathbf{F}(t_2) \rangle \right]. \tag{4.3}$$

Typically, the correlation function for the fluctuating force vanishes for times separated by much more than some correlation time, t_c . In this case, the contribution of the fluctuating force to $\langle \mathbf{v}^2 \rangle$ grows linearly in time.

We wish to consider the motion of a test particle in a weak gravitational field, so

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad (4.4)$$

where $|h_{\mu\nu}| \ll 1$. We further assume that the particle's motion is non-relativistic. However, we make no assumptions concerning the relative magnitudes of the stress tensor components, in contrast to the usual Newtonian limit where one assumes that T_{00} is large compared to all other components. The proper time interval along the particle's world line is

$$d\tau^{2} = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{00} dt^{2} + g_{ij} dx^{i} dx^{j}$$

= $dt^{2} (g_{00} + 2g_{0i} v^{i} + g_{ij} v^{i} v^{j}),$ (4.5)

where $v^i \equiv dx^i/dt$ is the velocity of the test particle. The dynamics of the test particle is determined by the variational principle

$$\delta \int (\frac{d\tau}{dt}) \, dt = 0. \tag{4.6}$$

Therefore, the Lagrangian of the test particle is

$$L = \frac{d\tau}{dt}$$

= $(-1 + h_{00} + 2h_{0i}v^{i} + \mathbf{v}^{2} + h_{ij}v^{i}v^{j})^{1/2}.$ (4.7)

Up to terms linear in $h_{\mu\nu}$, the Lagrangian is

$$L = -1 + \frac{1}{2}h_{00} + h_{0i}v^{i} + \frac{1}{2}\mathbf{v}^{2} + \frac{1}{2}h_{ij}v^{i}v^{j}.$$
(4.8)

The curvature tensor is related to $h_{\mu\nu}$ through

$$R_{\mu\nu} = \frac{1}{2} (h_{\nu\alpha}{}^{,\alpha}{}_{,\mu} + h_{\mu\alpha}{}^{,\alpha}{}_{,\nu} - \partial_{\rho}\partial^{\rho} h_{\mu\nu} - h_{,\mu\nu}), \qquad (4.9)$$

where $h = h^{\alpha}{}_{\alpha}$. In the Lorentz-Hilbert gauge,

$$\bar{h}_{\mu\nu}{}^{,\nu} \equiv (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\alpha}{}_{\alpha}){}^{,\nu} = 0, \qquad (4.10)$$

 $h_{\mu\nu}$ can be determined by the gravitational source through the gravitational field equations

$$\partial_{\rho}\partial^{\rho}h_{\mu\nu} = -16\pi G_N \bar{T}_{\mu\nu} \equiv -16\pi G_N \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}\right).$$
(4.11)

If we separate the spatial and temporal components, we have

$$\partial_{\rho}\partial^{\rho}h_{00} = -8\pi G_N \left(T_{00} + T^i_{\ i}\right),\tag{4.12}$$

$$\partial_{\rho}\partial^{\rho}h_{0i} = -16\pi G_N T_{0i}, \qquad (4.13)$$

and

$$\partial_{\rho}\partial^{\rho}h_{ij} = -16\pi G_N \left(T_{ij} + \frac{1}{2}\delta_{ij}T_{00} - \frac{1}{2}\delta_{ij}T^k{}_k\right).$$
(4.14)

Define generalized potentials, Φ , Ψ , ζ_i , and ξ_{ij} as linear combinations of the metric perturbations so that

$$h_{00} = 2\Phi + 2\Psi, \tag{4.15}$$

$$h_{0i} = \zeta_i, \tag{4.16}$$

$$h_{ij} = \xi_{ij} - 2\Phi\delta_{ij} + 2\Psi\delta_{ij}. \tag{4.17}$$

The potentials satisfy

$$\partial_{\rho}\partial^{\rho}\Phi = -4\pi G_N T_{00},\tag{4.18a}$$

$$\partial_{\rho}\partial^{\rho}\Psi = -4\pi G_N T^i{}_i, \qquad (4.18b)$$

$$\partial_{\rho}\partial^{\rho}\zeta_{i} = -16\pi G_{N}T_{0i}, \qquad (4.18c)$$

and

$$\partial_{\rho}\partial^{\rho}\,\xi_{ij} = -16\pi G_N T_{ij}.\tag{4.18d}$$

The Lagrangian of the test particle is

$$L = -1 + (\Phi + \Psi) + \zeta_i v^i + \frac{1}{2} \mathbf{v}^2 \left(1 - 2\Phi + 2\Psi\right) + \frac{1}{2} \xi_{ij} v^i v^j.$$
(4.19)

The Euler-Lagrange equation derived from this Lagrangian, after discarding terms higher than linear in the velocity or its first derivatives, is

$$\partial_i \Phi + \partial_i \Psi + \partial_i (\zeta_i v^j) - \dot{\zeta}_i + \dot{v}_i (1 - 2\Phi + 2\Psi) + v_i (-2\dot{\Phi} + 2\dot{\Psi}) + \dot{\xi}_{ij} v^j + \xi_{ij} \dot{v}^j = 0.$$
(4.20)

We can now solve for the time derivative of velocity and notice that in the weak field limit

$$(1 - 2\Phi + 2\Psi + \xi_{ij})^{-1} \approx 1 + 2\Phi - 2\Psi - \xi_{ij}, \qquad (4.21)$$

 \mathbf{SO}

$$\dot{v}_i = \dot{\zeta}_i - \partial_i \Phi - \partial_i \Psi + (2\dot{\Phi}\,\delta_{ij} - 2\dot{\Psi}\,\delta_{ij} - \partial_i \zeta_j - \dot{\xi}_{ij})\,v_j - \zeta_j(\partial_i\,v_j). \tag{4.22}$$

To lowest order

$$v_i = \zeta_i - \int (\partial_i \Phi(x)) dt - \int (\partial_i \Psi(x)) dt.$$
(4.23)

Equations (4.18a- 4.18d) may be solved in terms of the retarded Green's function,

$$G_{ret}(x) = -\frac{1}{4\pi |\mathbf{x}|} \,\,\delta(|\mathbf{x} - t|),\tag{4.24}$$

to yield

$$\Phi(t, \mathbf{x}) = G_N \int d^3x' \frac{T_{00}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \qquad (4.25a)$$

$$\Psi(t, \mathbf{x}) = G_N \int d^3x' \frac{T^i{}_i(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \qquad (4.25b)$$

$$\zeta_i(t, \mathbf{x}) = 4G_N \int d^3x' \frac{T_{0i}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \qquad (4.25c)$$

and

$$\xi_{ij}(t, \mathbf{x}) = 4G_N \int d^3x' \frac{T_{ij}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$
(4.25d)

The expression for the velocity, Eq. (4.23), now becomes

$$v_{i}(t) = 4G_{N} \int d^{3}x' \frac{T_{0i}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - G_{N} \partial_{i} \int d^{3}x' \int^{t} dt' \frac{T_{00}(t' - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') + T^{i}{}_{i}(t' - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$
(4.26)

The mean square of the i-component of the velocity is then

$$\langle v_i^2(t) \rangle = 16C_{0i0i}(t,t) + \partial_i^2 \int^t dt' dt'' C_{0000}(t',t'') + \partial_i^2 \int^t dt' dt'' C_{j}^{j}{}_k^k(t',t'') - 8\partial_i \int^t dt' C_{0i00}(t,t') - 8\partial_i \int^t dt' C_{0ij}{}^j(t,t') - 8\partial_i^2 \int^t dt' dt'' C_{00j}{}^j(t',t''), \qquad (4.27)$$

where

$$C_{\mu\nu\alpha\beta}(t',t'') \equiv G_N^2 \int d^3x' d^3x'' \frac{\langle :T_{\mu\nu}(t'-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') T_{\alpha\beta}(t''-|\mathbf{x}-\mathbf{x}''|,\mathbf{x}''):\rangle}{|\mathbf{x}-\mathbf{x}'| |\mathbf{x}-\mathbf{x}''|}.$$
 (4.28)

Note that in Eq. (4.27), the index *i* is not summed, but all other repeated indices are summed.

This expression is rather cumbersome to evaluate explicitly, but we can make order of magnitude estimates. Note that it is of the general form of Eq. (4.3), where the correlation function of the fluctuating force is of order

$$\langle FF \rangle \approx m^2 C,$$
 (4.29)

where C is a typical component of C_{ijkl} . If t_c is the correlation time, then we obtain the estimate

$$\langle v_i^2(t) \rangle \approx C t_c t.$$
 (4.30)

To be specific, let us consider a test particle moving in the fluctuating gravitational field produced by the Casimir vacuum. We assume that the particle is initially shot parallel to and midway between two plates. Classically, the particle will continue to move midway between the plates, as the gravitational force vanishes by symmetry. However, it is subjected to a fluctuating force which tends to cause it to drift toward one side or the other. Since the energy density is of the order of $1/L^4$, the characteristic force on the test particle is of order $G_N m/L^3$, so $C \approx G_N^2/L^6$. Furthermore, the correlation time is of order L, the only length scale in the problem. Therefore,

$$\langle v_i^2(t) \rangle \approx \left(\frac{L_P}{L}\right)^4 \frac{t}{L},$$
(4.31)

where L_P is the Planck length. Thus the transverse velocity of the test particle undergoes a random walk, with the rms transverse velocity growing as the square root of time. We have of course restricted our attention to a particular initial condition for the test particles, but a similar analysis could be performed for any other initial condition. The resulting probabilistic statements about such quantities as $\langle v_i^2(t) \rangle$ sum up all that we can know about a fluctuating gravitational field, such as that whose source is the Casimir vacuum energy.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have argued that the semiclassical theory of gravitation based upon Eq.(1.1) is an approximation which assumes that the fluctuations in $T_{\mu\nu}$ are small. This assumption can fail far away from the Planck scale. The approximation is valid for systems described by coherent states, but breaks down for more general quantum states. In particular, states of a quantized field in which the mean energy density is negative seem to exhibit large energy density fluctuations. This was illustrated in our model calculations for squeezed states and for the Casimir vacuum state. Thus the gravitational field of such a system is not described by a fixed classical metric, but rather by a fluctuating metric.

Because the operational significance of a gravitational field lies in its effects upon test particles, the meaning of a fluctuating gravitational field is given by a statistical description of ensembles of test particles. This notion was developed in Sec. IV, where we proposed a Langevin-type equation to determine such quantities as the mean squared velocity of the test particles.

The analysis in this paper was restricted to a flat spacetime background. In this case, the expectation values of quartic operators which appear in the dimensionless measure of the fluctuations, Δ , or in the Langevin equation, Eq.(4.27), may be defined in terms of normal ordered products. The generalization of this analysis to curved spacetime backgrounds will

require a renormalization procedure for quartic operator products in such spacetimes, which has not yet been developed. Note that an alternative approach to fluctuations is to discuss only averaged quantities. If we were to average fields over finite space or time intervals, then we could define a quantity similar to Δ without invoking normal ordering. This approach is used by Barton [10] in his treatment of the fluctuations of the Casimir force. In the present context, this procedure suffers from the defect that it introduces an arbitrary length or time scale into the problem. If, for example, one wished to obtain an equation of the form of Eq.(4.3) using averaged rather than normal-ordered quantities, there would be an inherent ambiguity in the resulting equation.

A topic of considerable current interest is the possibility of causality violation by a Lorentzian wormhole. Morris, Thorne, and Yurtsever [5] have shown how such a wormhole might in principle be constructed using the Casimir vacuum energy. A violation of the weak energy condition, and hence negative energy densities, is essential for the existence of a wormhole. Hochberg and Kephart [11] have suggested squeezed states as a possible source for the negative energy, but an explicit wormhole model using squeezed states has not yet been constructed. All of the literature on wormhole models assumes the semiclassical gravity theory. However, in light of the considerations in the present paper, it seems to be important to include the possible effects of metric fluctuations. Whether they are sufficiently large to prevent the creation of a wormhole or its use to violate causality is not clear. Another question of interest is whether a breakdown of the semiclassical theory provides an alternative explanation as to why negative energy fluxes cannot lead to violations of cosmic censorship [4].

ACKNOWLEDGMENTS

The work was supported in part by the National Science Foundation under Grant PHY-9208805.

REFERENCES

- [1] L. H. Ford, Ann. Phys. **144**, 238 (1982).
- [2] H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965).
- [3] L. H. Ford, Proc. R. Soc. Lond. A **364**, 227 (1978).
- [4] L. H. Ford and T. A. Roman, Phys. Rev. D 41, 3662(1990); 46, 1328 (1992).
- [5] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
- [6] L. H. Ford, Phys. Rev. D 41, 3662 (1990).
- [7] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, Phys. Rev. Lett. 57, 2520 (1986).
- [8] C. M. Caves, Phys. Rev. D 23, 1693 (1981).
- [9] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wet. **51**, 793 (1948).
- [10] G. Barton, J. Phys. A **24**, 991 & 5533 (1991).
- [11] D. Hochberg and T. W. Kephart, Phys. Lett. B 251, 349 (1991).

FIGURES

FIG. 1. The energy density ρ is plotted as a function of the squeeze parameter r and of $s = |\alpha|$. Here $\gamma = \delta = 0$ and $\theta = \pi/2$. When s >> r, the state is close to a coherent state, and $\rho > 0$. However, when $s \ll r$, the energy density is negative.

FIG. 2. Here Δ is plotted for the same range and choices of parameters as in Fig. 1. Note that $\Delta \ll 1$ only when $s \gg r$, the region of classical behavior. Otherwise, $\Delta \approx 1$.

FIG. 3. The energy density ρ is plotted as a function of the squeeze parameter r and of γ , which is the phase of α . Here $\delta = 0$, $s = \frac{1}{2}$, and $\theta = \pi/2$.

FIG. 4. Here Δ is plotted for the same range and choices of parameters as in Fig. 3. Again $\Delta \ll 1$ only when $r \ll 1$.